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A Nonlocal Metric Formulation of MOND

M. E. Soussa[†] and R. P. Woodard[‡]

Department of Physics
University of Florida
Gainesville, FL 32611 USA

ABSTRACT

We study a class of nonlocal, but causal, covariant and conserved field equations for the metric. Although nonlocal, these equations do not seem to possess extra graviton solutions in weak field perturbation theory. Indeed, the equations reduce to those of general relativity when the Ricci scalar vanishes throughout spacetime. When a static matter source is present we show how these equations can be adjusted to reproduce Milgrom's Modified Newtonian Dynamics in the weak field regime, while reducing to general relativity for strong fields. We compute the angular deflection of light in the weak field regime and demonstrate that it is the same as for general relativity, resulting in far too little lensing if no dark matter is present. We also study the field equations for a general Robertson-Walker geometry. An interesting feature of our equations is that they become conformally invariant in the MOND limit.

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[†] e-mail: soussam@phys.ufl.edu
[‡] e-mail: woodard@phys.ufl.edu

1 Introduction

Milgrom's Modified Newtonian Dynamics (MOND) [1] is an empirical alternative to invoking dark matter as an explanation for the motions of cosmological systems which are subject to very small accelerations. In one formulation the force $\vec{F}_{\rm Newt}$ on a point mass m is given by Newtonian gravity, but the second law becomes nonlinear in the acceleration $\vec{a}$,

$$\vec{F}_{\rm Newt} = m\mu\Big(\frac{a}{a_0}\Big)\vec{a} \qquad \text{where} \qquad \mu(x) = \begin{cases} 1 & \forall x \gg 1 \; , \\ x & \forall x \ll 1 \; . \end{cases} \tag{1}$$

The numerical value of a_0 has been determined by fitting to the rotation curves of nine well-measured galaxies [2],

$$a_0 = (1.20 \pm .27) \times 10^{-10} \ {\rm m/s}^2 \; . \tag{2}$$

Because the observed stars and gas contain constituents subject to much larger internal accelerations[1] it is preferable to view MOND as a modification of the gravitational force at low accelerations. That is, the second law takes its traditional form $\vec{F}_{\rm MOND} = m\vec{a}$, but the actual force is the following nonlinear function of the force predicted by Newtonian gravity,

$$\vec{F}_{\rm MOND} = f\Big(\frac{F_{\rm Newt}}{ma_0}\Big)\vec{F}_{\rm Newt} \qquad \text{where} \qquad f(x) = \begin{cases} 1 & \forall x \gg 1 \; , \\ x^{-\frac{1}{2}} & \forall x \ll 1 \; . \end{cases} \tag{3}$$

One consequence of MOND is that the velocities of particles in progressively more distant circular orbits approach a constant v_∞ which depends only upon a galaxy's mass [3]. For large radius r the Newtonian force due to a galaxy of mass M is $F_{\rm Newt} = GMm/r^2$. Since $F_{\rm Newt}/m$ must eventually fall below a_0, relation (3) implies that the asymptotic force law is $F_{\rm MOND} = m\sqrt{a_0 GM}/r$. Applying the second law gives,

$$\frac{\sqrt{a_0 GM}}{r} = \frac{v_\infty^2}{r} \qquad \Longrightarrow \qquad a_0 GM = v_\infty^4 \; . \tag{4}$$

With no dark matter a galaxy's luminosity L should be a constant times its mass, although the constant will depend upon the type of galaxy. Thus one expects $L \sim v_\infty^4$, which is the observed Tully-Fisher relation [4].

[1]For example, were we to think of neutral Hydrogen as a classical planetary system, the proton would be subject to an acceleration of about 10^{19} m/s^2 about the atomic barycenter!

MOND provides an excellent fit to the rotation curves of all known types of galaxies, using only the measured distributions of gas and stars and fitted mass-to-luminosity ratios for gas and stars. The recent review by Sanders and McGaugh [5] summarizes the increasingly compelling data and lists the primary sources. It is significant that the fitted mass-to-luminosity ratios are not unreasonable [6]. Especially significant is that MOND agrees in detail even with low surface brightness galaxies [7, 8], objects for which the MOND regime ($a \lesssim a_0$) applies throughout and for which no detailed measurements had been made when MOND was proposed. On the other hand, some dark matter must be invoked to explain the temperature and density profiles of large galaxy clusters [9]. (It it has been suggested that eV-mass neutrinos might provide this without affecting the galactic results [10].) And a very recent analysis of data from the Sloan Digital Sky Survey asserts that satellites of isolated galaxies violate MOND when care is taken to exclude interlopers [11].

The successes of MOND might be a numerical coincidence [12]. They may result instead from galaxy formation and evolution flowing towards some yet-unrecognized attractor solution through conventional physics. Or they might signal a real modification of gravity in the regime of very low acceleration. Although the issue cannot be decided at this time, the number and quality of relevant observations which are technically feasible [13] indicate that it may be resolved in the near future. If MOND could be convincingly embedded in a larger, metric formulation of gravity, that theory could *already* be tested against a wide variety of data from lensing [14, 15], cosmology [16], and structure formation [17]. As it is, one must make assumptions, whereupon comparison with observation is as much a test of these assumptions as it is of MOND. It is therefore opportune to consider what sort of fundamental theory might reduce to MOND in the appropriate limit.

There is a satisfactory nonrelativistic potential formulation for the relation between the MOND force $\vec{F}_{\rm MOND} = -\vec{\nabla}\phi$ and the mass density ρ_m. It is given by the Lagrangian of Bekenstein and Milgrom [18],

$$\mathcal{L} = -\rho_m \phi - \frac{a_0^2}{8\pi G} F\Bigl(\frac{\|\vec{\nabla}\phi\|^2}{a_0^2}\Bigr) \qquad \text{where} \qquad \mu(x) = F'(x^2) \ . \tag{5}$$

This is very important because it establishes that MOND conserves energy, momentum and angular momentum. However, it does not extend the theory sufficiently to test lensing and cosmology. A generally coordinate invariant, scalar-metric extension exists but it contains dynamical scalar degrees of freedom which can propagate acausally [18].

We wish here to consider a different possibility. Suppose that general relativity really is the fundamental theory of gravity, but that its effective action contains large quantum corrections from infrared virtual particles. Weinberg showed that infrared effects in quantum gravity are no stronger than those of QED for the case of zero cosmological constant [19]. However, a nonzero cosmological constant would preserve the graviton's masslessness while subjecting it to interactions of canonical dimension three. Recall that infrared effects become stronger as massless particles are coupled with lower dimension interactions, and that they are already nonperturbatively strong for massless gluons with the dimension four coupling of QCD.

These considerations are the basis for a daring proposal to simultaneously resolve the (old) cosmological constant problem [20, 21] and provide a natural model of inflation in which scalars play no part. The idea [22] is that the bare cosmological constant is actually GUT-scale, which leads to an initial period of inflation during the early universe. What brings inflation to an end is the gravitational attraction between the ever-increasing numbers of infrared gravitons ripped out of the vacuum by the rapid expansion. One can follow this process as long as the slowing is weak, and explicit computations confirm that the effect must eventually become nonperturbatively strong [23].

If this proposal is correct, the post-inflationary universe would be described by the nonlocal effective action which prevails after the breakdown of perturbation theory. We cannot compute reliably in this regime, but nothing prevents one from making guesses about the form of this effective action. The simplest class of guesses which give a plausible end for inflation involve acting the inverse covariant d'Alembertian on the Ricci scalar [24]. We will refer to this as *the small potential*,

$$\varphi[g] \equiv \frac{1}{\square} R \qquad \text{where} \qquad \square \equiv \frac{1}{\sqrt{-g}} \partial_\mu \Big(\sqrt{-g} g^{\mu\nu} \partial_\nu \Big) \; . \tag{6}$$

(We use a spacelike metric with $R_{\mu\nu} \equiv \Gamma^\rho{}_{\nu\mu,\rho} - \Gamma^\rho{}_{\rho\mu,\nu} + \Gamma^\rho{}_{\rho\sigma} \Gamma^\sigma{}_{\nu\mu} - \Gamma^\rho{}_{\nu\sigma} \Gamma^\sigma{}_{\rho\mu}$.) This paper will not consider inflation or the cosmological constant. We shall rather explore embedding MOND in a nonlocal Lagrangian of the form,

$$\mathcal{L} = \frac{c^4}{16\pi G} \Big[R + c^{-4} a_0^2 \mathcal{F}\Big(c^4 a_0^{-2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} \Big) \Big] \sqrt{-g} \; . \tag{7}$$

A desirable feature of this class of models is that it involves only the metric. Although the field equations are nonlocal, they do not seem to possess additional graviton solutions in weak field perturbation theory. To see this, expand the metric about flat space as usual,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; . \tag{8}$$

It is an elementary exercise to show that the Ricci scalar is,

$$R = h^{\mu\nu}{}_{,\mu\nu} - h^{\mu}{}_{\mu}{}^{,\nu}{}_{\nu} + O(h^2) \; , \tag{9}$$

where graviton indices are raised and lowered using the Lorentz metric as usual. Now impose de Donder gauge,

$$h^{\mu}{}_{\nu,\mu} - \frac{1}{2} h^{\mu}{}_{\mu,\nu} = 0 \; , \tag{10}$$

to show that the small potential is local in the weak field limit,

$$\varphi[\eta + h] = -\frac{1}{2} h^{\mu}{}_{\mu} + O(h^2) \; . \tag{11}$$

Since the Lagrangian depends upon the first derivative of φ, this theory cannot acquire higher derivative solutions in weak field perturbation theory. Indeed, all solutions to the source-free Einstein equations must be solutions to this theory because they have $R = 0$ throughout spacetime, which implies $\varphi = 0$ as well. The correction term in (7) changes only gravity's response to sources, without adding new weak field dynamical degrees of freedom. So these models seem worthy of closer study.

In section 2 we derive the form a static and spherically symmetric metric must take to reproduce galactic rotation curves in the MOND regime. We also work out the deflection of light in the weak field limit. In section 3 we employ a formal shortcut to obtain the causal field equations which would come from a Schwinger-Keldysh effective action [25]. We also explicitly demonstrate conservation. The nonlocal field equations are specialized to a static, spherically symmetric geometry in section 4. We first prove, for general interpolating function $\mathcal{F}(x)$, that any MOND terms must drop out of the formula for the deflection of light. This is the decisive point of our analysis because the existing lensing data can only be made consistent with the absence of dark matter if MOND terms contribute [15]. No member of this class of models can therefore represent a phenomenologically viable extension of MOND. However, it is possible to choose the interpolating function $\mathcal{F}(x)$ to reproduce MOND, and we derive the required form. Section 5 explores the model's impact on cosmology, more to illustrate what can be done with a complete theory than because this model is viable. In section 6 we discuss more general models which might avoid the lensing disaster.

2 Phenomenological constraints

MOND was developed as an alternative to halos of dark matter surrounding nearly static, cosmological sources. Far from such a system one would expect the asymptotic geometry to be spherically symmetric and static:

$$ds^2 \equiv g_{\mu\nu}(x)dx^\mu dx^\nu = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \; . \tag{12}$$

We first show that rotation curves fix $B(r)$ in the MOND limit but tell us nothing about $A(r)$. We then discuss the magnitude of the effect. The section closes by working out the angular deflection of light for a general class of weak field limits.

Let $\chi^\mu(t)$ be the worldline of a test particle moving in the geometry described by (12). If only the gravitational force is significant the particle's worldline obeys the geodesic equation:

$$\ddot{\chi}^\mu(t) + \Gamma^\mu{}_{\rho\sigma}\Bigl(\chi(t)\Bigr)\dot{\chi}^\rho(t)\dot{\chi}^\sigma(t) = 0 \; . \tag{13}$$

The nonzero connection components derived from (12) are,

$$\begin{aligned}
&\Gamma^t{}_{tr} = \frac{B'}{2B} \; , \; \Gamma^r{}_{tt} = \frac{B'}{2A} \; , \; \Gamma^r{}_{rr} = \frac{A'}{2A} \; , \; \Gamma^r{}_{\theta\theta} = -\frac{r}{A} \; , \; \Gamma^r{}_{\phi\phi} = -\frac{r}{A}\sin^2(\theta) \; , \\
&\Gamma^\theta{}_{\theta r} = \frac{1}{r} \; , \; \Gamma^\phi{}_{\phi r} = \frac{1}{r} \; , \; \Gamma^\theta{}_{\phi\phi} = -\sin(\theta)\cos(\theta) \; , \; \Gamma^\phi{}_{\phi\theta} = \cot(\theta) \; .
\end{aligned} \tag{14}$$

Now specialize to the case of circular motion,

$$\Bigl(\chi^t, \chi^r, \chi^\theta, \chi^\phi\Bigr) = \Bigl(ct, r, \frac{\pi}{2}, \phi(t)\Bigr) \; . \tag{15}$$

With our connection (14) the $\mu = t$ and $\mu = \theta$ components of the geodesic equation are tautologies. The $\mu = \phi$ component just says $\dot{\phi}$ is constant. Only the $\mu = r$ component of the geodesic equation is nontrivial,

$$\frac{B'}{2A} - \frac{r}{A}\frac{\dot{\phi}^2}{c^2} = 0 \; . \tag{16}$$

Note that $A(r)$ factors out! For circular orbits the velocity has the Euclidean relation to the angular velocity, $v = r\dot{\phi}$. In the MOND limit v^2 approaches the constant $v_\infty^2 = \sqrt{a_0 GM}$, so the MOND limit for $B(r)$ must obey,

$$B'(r) \longrightarrow \frac{2}{r}\sqrt{\frac{a_0 GM}{c^4}} \; . \tag{17}$$

It is well at this point to consider the size of things. A large galaxy might have a mass in stars and gas of $M \sim 10^{11} \times M_\odot \sim 10^{41}$ kg. Such a galaxy would enter the MOND regime at a radius of about,

$$R_{\rm gal} \sim \sqrt{\frac{GM}{a_0}} \sim 10^{20} \text{ m} \; . \tag{18}$$

For weak fields we can write,

$$A(r) = 1 + a(r) \qquad , \qquad B(r) = 1 + b(r) \; , \tag{19}$$

where $|a(r)| \ll 1$ and $|b(r)| \ll 1$. A useful phenomenological ansatz for the asymptotic behavior of the weak fields is,

$$a(r) \longrightarrow \delta_1 \frac{GM}{c^2 r} + \epsilon_1 \sqrt{\frac{a_0 GM}{c^4}} \; , \; b(r) \longrightarrow \delta_2 \frac{GM}{c^2 r} + \epsilon_2 \sqrt{\frac{a_0 GM}{c^4}} \ln\Bigl(\frac{r}{R_{\rm gal}}\Bigr) \; . \tag{20}$$

We have just seen that MOND predicts $\epsilon_2 = 2$ but says nothing about the other parameters. With just the isolated galaxy, general relativity gives $\delta_1 = -\delta_2 = 2$ and $\epsilon_1 = \epsilon_2 = 0$. If an isothermal halo of dark matter is added, whose density is chosen to reproduce $v_\infty^2 = \sqrt{a_0 GM}$, general relativity gives $\delta_1 = -\delta_2 = \epsilon_1 = \epsilon_2 = 2$.

One might worry that the logarithmic growth of $b(r)$ in (20) must eventually invalidate the weak field approximation but this is not a practical concern. For the large galaxy considered previously the small parameters multiplying the δ's and ϵ's are,

$$\frac{GM}{c^2 r} \sim \frac{R_{\rm gal}}{r} \times 10^{-6} \qquad , \qquad \sqrt{\frac{a_0 GM}{c^4}} \sim 10^{-6} \; . \tag{21}$$

The difference $b(r)$ from the onset of MOND all the way to the current horizon ($R_{\rm hor} \sim 10^{26}$ m) is,

$$b(R_{\rm hor}) - b(R_{\rm gal}) \sim -\delta_2 \times 10^{-6} + \epsilon_2 \times 10^{-5} \; . \tag{22}$$

The weak field regime is therefore applicable throughout the Hubble volume.

Another numerical fact worthing noting is that the natural length associated with the MOND acceleration a_0 is larger than the Hubble radius,

$$\frac{c^2}{a_0} \sim 10^{27} \text{ m} . \tag{23}$$

This has the important consequence that ra_0/c^2 is negligible on galaxy and galactic cluster scales, so that powers of r do not necessarily distinguish "large" and "small" terms. Consider, for example, two possible contributions to the weak fields,

$$\frac{a_0 GM}{c^4} \ln\Bigl(\frac{r}{R_{\rm gal}}\Bigr) \ll \frac{GM}{c^2 r} . \tag{24}$$

The left hand term is negligible with respect to the right hand side, even though the latter falls off with r whereas the former actually grows.

We can now consider what MOND says about lensing. The angular deflection of light can be expressed in terms of the turning point R_0,

$$\Delta\phi = 2\int_{R_0}^{\infty} \frac{dr}{r} \frac{A^{\frac{1}{2}}(r)}{\sqrt{\left(\frac{r}{R_0}\right)^2 \frac{B(R_0)}{B(r)} - 1}} - \pi . \tag{25}$$

This can be expanded in powers of the weak fields and then simplified using the change of variables $r = R_0 \sec(\theta)$,

$$\Delta\phi = 2\int_{R_0}^{\infty} \frac{dr}{r} \frac{1}{\sqrt{\left(\frac{r}{R_0}\right)^2 - 1}} \left\{ 1 + \frac{a(r)}{2} - \frac{1}{2}\frac{b(R_0) - b(r)}{1 - \left(\frac{R_0}{r}\right)^2} + \dots \right\} - \pi, \tag{26}$$

$$= \int_0^{\frac{\pi}{2}} d\theta \left\{ a\Bigl(R_0 \sec(\theta)\Bigr) - \csc^2(\theta)\Bigl[b\Bigl(R_0\Bigr) - b\Bigl(R_0 \sec(\theta)\Bigr)\Bigr] + \dots \right\}. \tag{27}$$

Substituting the general ansatz (20) gives,

$$\Delta\phi = \Bigl(\delta_1 - \delta_2\Bigr)\frac{GM}{c^2 R_0} + \Bigl(\epsilon_1 + \epsilon_2\Bigr)\frac{\pi}{2}\sqrt{\frac{a_0 GM}{c^4}} + \dots . \tag{28}$$

Without dark matter, general relativity ($\delta_1 = -\delta_2 = 2$ and $\epsilon_1 = \epsilon_2 = 0$) gives too little deflection at large R_0 to be consistent with the frequency of lensing by galaxies. General relativity with an isothermal halo of dark matter ($\delta_1 = -\delta_2 = \epsilon_1 = \epsilon_2 = 2$) is consistent with the existing data [15]. For MOND to be similarly consistent with the data requires the sum $(\epsilon_1 + \epsilon_2)$ to be positive and of order one.

3 The nonlocal field equations

The ostensible purpose of this section is to derive the field equations associated with (7). We are not going to quite do that for the very good reason that one does not get causal field equations by varying a temporally nonlocal action. To see this, consider the quadratic part of the effective action for a real scalar field $\phi(x)$ in flat spacetime,

$$\Gamma[\phi] = \frac{1}{2} \int d^4y \phi(y) [\partial^2 - m^2] \phi(y) - \frac{1}{2} \int d^4y \int d^4z \phi(y) M^2(x;y) \phi(z) \; . \quad (29)$$

Here $M^2(y;z)$ is the scalar self-mass-squared, which is symmetric under interchange of y^μ and z^μ. Taking the variational derivative and ignoring surface terms gives,

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = [\partial^2 - m^2] \phi(x) - \int d^4x' M^2(x;x') \phi(x') \; . \quad (30)$$

If $M^2(x;x') \neq 0$ for some $x'^\mu = x^\mu - \Delta^\mu$ in the past of x^μ then it is also nonzero for $x'^\mu = x^\mu + \Delta^\mu$ in the future of x^μ. Another embarrassing property of the conventional effective action field equations is that their solutions are not typically real even if the associated field operators are Hermitian. This is because the effective action field equations are obeyed by in-out matrix elements of the field operators, and the "in" state may not evolve into the same "out" state.

Both causality and reality can be enforced by employing the field equations of the Schwinger-Keldysh effective action [25]. These are obeyed by the expectation values of the field operators in the presence of the same state, so Hermiticity of the operator implies reality of the solution. In the Schwinger-Keldysh formalism there are two background fields: $\phi_+(x)$ for the evolution forward from the initial state and $\phi_-(x)$ for the reverse evolution back to it.

The quadratic part of the scalar action has the form,

$$\Gamma[\phi_+;\phi_-] = \frac{1}{2}\int d^4y \phi_+(y)[\partial^2 - m^2]\phi_+(y) - \frac{1}{2}\int d^4y \phi_-(y)[\partial^2 - m^2]\phi_-(y) + \frac{1}{2}\int d^4y \int d^4z \Big\{\phi_+(y) M^2_{++}(y;z)\phi_+(z) + \phi_+(y) M^2_{+-}(y;z)\phi_-(z) + \phi_-(y) M^2_{-+}(y;z)\phi_+(z) + \phi_-(y) M^2_{--}(y;z)\phi_-(z)\Big\} \; , \qquad (31)$$

where $M^2_{++}(y;z)$ and $M^2_{--}(y;z)$ are symmetric, and $M^2_{+-}(y;z) = M^2_{-+}(z;y)$. The various $\pm$ self-mass-squared functions are related by simple rules which alter the $i\epsilon$ terms in propagators and change the sign of some vertices [25]. The Schwinger-Keldysh field equations are obtained by varying with respect to either ϕ_+ or ϕ_-, and then equating the two fields after the variation [25],

$$\frac{\delta\Gamma[\phi_+;\phi_-]}{\delta\phi_+(x)}\Big\vert_{\phi_\pm = \phi} = [\partial^2 - m^2]\phi(x) - \int d^4x' [M^2_{++}(x;x') + M^2_{+-}(x;x')]\phi(x') \; . \qquad (32)$$

It turns out that $M^2_{+-}(x;x')$ is exactly equal and opposite to $M_{++}(x;x')$ whenever x'^μ is not in or on the past light-cone of x^μ. When x'^μ is in the past of x^μ the two terms are complex conjugates, which ensures reality.[2]

Since we do not actually derive either the conventional effective action or the Schwinger-Keldysh version, we will simply employ a trick to extract causal field equations from (7). For the purposes of this paper one may as well regard these equations — rather than (7) — as defining the model. The trick is to act the nonlocal operators backwards whenever they would ordinarily act upon the variation. For example, when $f[g]$ is any functional of the metric we write,

$$f[g] \frac{\delta\varphi[g]}{\delta g^{\mu\nu}} = f[g] \Big\{ - \frac{1}{\square} \frac{\delta\square}{\delta g^{\mu\nu}} \frac{1}{\square} R + \frac{1}{\square} \frac{\delta R}{\delta g^{\mu\nu}} \Big\} \; , \qquad (33)$$

$$\longrightarrow \Big\{ - \frac{\delta\square}{\delta g^{\mu\nu}} \varphi + \frac{\delta R}{\delta g^{\mu\nu}} \Big\} \frac{1}{\square} f[g] \; . \qquad (34)$$

It is useful to recall the standard result for varying the Ricci scalar,

$$\frac{\delta R(y)}{\delta g^{\mu\nu}(x)} = \Big[R_{\mu\nu}(y) + D_\mu D_\nu - g_{\mu\nu}(y) \square \Big] \delta^4(y - x) \; . \qquad (35)$$

[2]Full details can be found in a recent paper on the vacuum polarization of scalar QED in a locally de Sitter background [26].

(Here D_μ is the covariant derivative operator.) We recall also the definition of the stress-energy tensor from the variation the matter action S_m,[3]

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \; . \tag{36}$$

Taking $16\pi G c^{-4}/\sqrt{-g}$ times the variation of our nonlocal action (7) — in the sense of our trick (34) — gives the following field equations,

$$\begin{aligned} 8\pi G c^{-4} T_{\mu\nu} &= 2[\Phi_{;\mu\nu} - g_{\mu\nu} \Box \Phi] + G_{\mu\nu}[1 - 2\Phi] \\ &+ \Bigl[g_{\mu\nu} \varphi^{,\rho} \Phi_{,\rho} - \varphi_{,\mu} \Phi_{,\nu} - \varphi_{,\nu} \Phi_{,\mu}\Bigr] + \varphi_{,\mu} \varphi_{,\nu} \mathcal{F}' - \frac{a_0^2}{2c^4} g_{\mu\nu} \mathcal{F} \; . \end{aligned} \tag{37}$$

The symbol $\Phi[g]$ in this and subsequent formulae denotes the *large potential*,

$$\Phi[g] \equiv \frac{1}{\Box} \Bigl(\varphi^{,\rho} \mathcal{F}' \Bigr)_{;\rho} \; . \tag{38}$$

Because we have not really derived (37) it is worth while to explicitly verify the important property of conservation. The covariant divergence of each term is,

$$2[\Phi_{;\mu\nu} - g_{\mu\nu} \Box \Phi]^{;\nu} = +2 R_\mu^{\ \nu} \Phi_{,\nu} \; . \tag{39}$$

$$\Bigl(G_{\mu\nu}[1 - 2\Phi] \Bigr)^{;\nu} = -2 R_\mu^{\ \nu} \Phi_{,\nu} + R \Phi_{,\mu} \; , \tag{40}$$

$$\Bigl[g_{\mu\nu} \varphi^{,\rho} \Phi_{,\rho} - \varphi_{,\mu} \Phi_{,\nu} - \varphi_{,\nu} \Phi_{,\mu}\Bigr]^{;\nu} = -\varphi_{,\mu} \Box \Phi - R \Phi_{,\mu} \; , \tag{41}$$

$$\Bigl(\varphi_{,\mu} \varphi_{,\nu} \mathcal{F}' \Bigr)^{;\nu} = +\varphi_{;\mu\nu} \varphi^{,\nu} \mathcal{F}' + \varphi_{,\mu} \Box \Phi \; , \tag{42}$$

$$\Bigl(-\frac{a_0^2}{2c^4} g_{\mu\nu} \mathcal{F} \Bigr)^{;\nu} = -\varphi_{;\mu\nu} \varphi^{,\nu} \mathcal{F}' \; , \tag{43}$$

and they do sum to zero. The equations (37) are manifestly covariant. If $1/\Box$ denotes the retarded Green's function they are also causal.

4 Spherically symmetric, static sources

This is the key section of the paper. We begin by working out the small potential (6) and the large potential (38) for a general spherically symmetric

[3] This should be local because only gravitons combine the properties (natural masslessness and the absence of conformal invariance) needed to produce a strong infrared effect during inflation [22].

and static metric (12). Then we give the two independent equations which derive from (37) for this geometry. This is valid for arbitrary $A(r) = 1 + a(r)$ and $B(r) = 1 + b(r)$. In the MOND regime one has $|a(r)| \ll 1$ and $|b(r)| \ll 1$, for which case we show that the equations can be simplified without making any assumption about the interpolating function $\mathcal{F}(x)$. In terms of the general weak field ansatz (20), these simplified equations prove that $\epsilon_1 + \epsilon_2 = 0$. It follows from section 2 that no model of the form (7) can be consistent with galaxy lensing. However, it is possible to choose the interpolating function $\mathcal{F}(x)$ so as to reproduce MOND's success with galactic rotation curves. The section closes by doing this.

The general spherically symmetric and static geometry (12,14) gives rise the following Ricci scalar,

$$R = -\frac{B''}{AB} + \frac{B'}{2AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{2}{rA}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{2}{r^2}\left(1 - \frac{1}{A}\right) . \tag{44}$$

In this geometry, and acting upon a function only of r, the covariant d'Alembertian reduces to,

$$\square = \frac{1}{r^2\sqrt{AB}}\frac{d}{dr}\left(r^2\sqrt{\frac{B}{A}}\frac{d}{dr}\right) . \tag{45}$$

The differential equation which defines the small potential therefore takes the form,

$$\left(r^2\sqrt{\frac{B}{A}}\varphi'\right)' = \left(r^2\sqrt{\frac{B}{A}}\Big[-\frac{B'}{B} + \frac{2}{r}(A-1)\Big]\right)' - r\sqrt{AB}\left(1 - \frac{1}{A}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right) . \tag{46}$$

Assuming the two parenthesized terms vanish at $r = 0$ we can write,

$$\varphi'(r) = -\frac{B'}{B} + \frac{2}{r}\Big(A - 1\Big) - \frac{1}{r^2}\sqrt{\frac{A}{B}}\int_0^r dr' r'\sqrt{AB}\Big(1 - \frac{1}{A}\Big)\left(\frac{A'}{A} + \frac{B'}{B}\right) . \tag{47}$$

The differential equation that defines the large potential is,

$$\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\Phi_{,\nu}\Big) = \partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\varphi_{,\nu}\mathcal{F}'\Big) \Longrightarrow \left(r^2\sqrt{\frac{B}{A}}\Phi'\right)' = \left(r^2\sqrt{\frac{B}{A}}\varphi'\mathcal{F}'\right)' . \tag{48}$$

Assuming again that the parenthesized terms vanish at $r = 0$ we can write,

$$\Phi'(r) = \varphi'(r)\mathcal{F}'\left(\frac{c^4\varphi'^2(r)}{a_0^2 A(r)}\right) \; . \tag{49}$$

And those of its second covariant derivatives we shall need are,

$$\Phi_{;tt} = -\frac{B'}{2A}\Phi' \; , \; \Phi_{;rr} = \Phi'' - \frac{A'}{2A}\Phi' \; , \; \Box\Phi = \frac{1}{A}\Big[\Phi'' + \frac{2}{r}\Phi' + \frac{1}{2}\Big(\frac{B'}{B} - \frac{A'}{A}\Big)\Big] \; . \tag{50}$$

In this geometry only the diagonal components of the field equations (37) are nontrivial. The $\theta\theta$ and $\phi\phi$ equations are proportional to one another, and conservation gives both from the tt and rr equations the same as for the stress-energy,

$$\frac{T_{\phi\phi}}{\sin^2(\theta)} = T_{\theta\theta} = \frac{r^3}{2A}\Big\{\frac{B'}{2B}\frac{A}{B}T_{tt} + \Big[\frac{d}{dr} + \frac{2}{r} - \frac{A'}{A} + \frac{B'}{2B}\Big]T_{rr}\Big\} \; . \tag{51}$$

The system can therefore be defined by its tt (times A/B) and rr equations,

$$\frac{8\pi GA}{c^4B}T_{tt} = 2\Phi'' + \frac{4}{r}\Phi' + \frac{A}{B}G_{tt}(1-2\Phi) + \frac{a_0^2}{2c^4}A\mathcal{F} - \frac{A'}{A}\Phi' - \varphi'\Phi' \; , \tag{52}$$

$$\frac{8\pi G}{c^4}T_{rr} = -\frac{4}{r}\Phi' + G_{rr}(1-2\Phi) - \frac{a_0^2}{2c^4}A\mathcal{F} - \frac{B'}{B}\Phi'. \tag{53}$$

The tt and rr components of the Einstein tensor are,

$$\frac{A}{B}G_{tt} = \frac{A'}{rA} + \Big(\frac{A-1}{r^2}\Big) \qquad , \qquad G_{rr} = \frac{B'}{rB} - \Big(\frac{A-1}{r^2}\Big) \; . \tag{54}$$

We now evaluate the derivative of the small potential (47) to leading order in the weak fields, $a(r)$ and $b(r)$,

$$\varphi' \longrightarrow \frac{2a}{r} - b' + \dots \tag{55}$$

One might worry that the integral in (47) contributes as well, but note that the integrand exactly vanishes for $A = B^{-1}$. This means that the integral cannot contribute much in the regime for which general relativity applies. The integrand is nonzero in the MOND regime, but it is also second order in the weak fields, a and b. We can therefore ignore this term altogether.

In the asymptotic regime we can assume that each derivative adds a factor of $1/r$. Hence $\varphi'(r)$ goes like $1/r$ times the small numbers $a(r)$ or $b(r)$. It follows that Φ'/r is much larger in magnitude than $\varphi'\Phi'$. By similar reasoning we recognize that Φ'/r and Φ'' dominate the other MOND corrections,

$$\left|\frac{1}{r}\Phi'\right| \sim \left|\Phi''\right| \gg \left|\varphi'\Phi'\right| \; , \; \left|\frac{A'}{A}\Phi'\right| \; , \; \left|\frac{B'}{B}\Phi'\right| \; , \; \left|\frac{a_0^2}{c^4}\mathcal{F}\right| \; . \tag{56}$$

In the weak field limit it seems reasonable to assume $T_{rr} = 0$ while still allowing a nonzero $A/B \, T_{tt} = \rho$. We still don't know how the two leading MOND terms (56) on the gravitational side of the equations compare with the leading terms from general relativity. Including both gives,

$$2\Phi'' + \frac{4}{r}\Phi' + \frac{a'}{r} + \frac{a}{r^2} + \ldots \;\; = \;\; \frac{8\pi G}{c^4}\rho(r) \tag{57}$$

$$-\frac{4}{r}\Phi' + \frac{b'}{r} - \frac{a}{r^2} + \ldots \;\; = \;\; 0 \; . \tag{58}$$

The first of these equations (57) can be integrated to give,

$$\frac{4}{r}\Phi' + \frac{2a}{r^2} + \cdots = \frac{K}{r^3} + \frac{16\pi G}{c^4 r^3}\int_{R_{\rm gal}}^{r} dr' r'^2 \rho(r') \; . \tag{59}$$

Adding (58) and (59) cancels the leading MOND corrections,

$$\frac{b'}{r} + \frac{a}{r^2} + \cdots = \frac{K}{r^3} + \frac{16\pi G}{c^4 r^3}\int_{R_{\rm gal}}^{r} dr' r'^2 \rho(r') \; . \tag{60}$$

Equation (60) is interesting because it has no dependence upon the still unknown interpolating function $\mathcal{F}(x)$. We can therefore use it to make statements about all models of the type (7). Under the assumption of no dark matter halos, the mass integral must eventually stop growing, in which case the left hand side falls off like $1/r^3$. So if $b'(r)$ goes like a constant times $1/r$ then $a(r)$ must go like minus the same constant. In terms of the generic ansatz (20) of section 2 we have just demonstrated that all models of the type (7) have $\epsilon_1 + \epsilon_2 = 0$. As discussed in section 2, this means that galaxies without halos of dark matter would give far too little lensing. These models do not give phenomenologically viable realizations of MOND.

It is still interesting to see if the interpolating function $\mathcal{F}(x)$ can be chosen to reproduce MOND rotation curves. For this purpose let us consider a sphere of radius R with very low, constant density,

$$\rho(r) = \frac{3Mc^2}{4\pi R^3}\theta(R-r) \; . \tag{61}$$

If the density is small enough the MOND regime prevails throughout, as in a low surface brightness galaxy. This means that (57) can be integrated all the way down to $r = 0$ to give,

$$2\Phi' + \frac{a}{r} + \cdots = \frac{8\pi G}{c^4 r^2} \int_0^r dr' r'^2 \rho(r') \; . \tag{62}$$

We can also use (55) to eliminate $b'(r)$ in $-r$ times (58),

$$4\Phi' + \varphi' - \frac{a}{r} + \cdots = 0 \; . \tag{63}$$

Now eliminate $a(r)$ by adding (62) and (63), and then use (49) to obtain an equation for the small potential,

$$\varphi' \Bigl[1 + 6\mathcal{F}'\Bigl(\frac{c^4 \varphi'^2}{a_0^2}\Bigr)\Bigr] + \cdots = \frac{8\pi G}{c^4 r^2} \int_0^r dr' r'^2 \rho(r') \; . \tag{64}$$

For $r > R$ the mass integral is constant,

$$\varphi' \Bigl[1 + 6\mathcal{F}'\Bigl(\frac{c^4 \varphi'^2}{a_0^2}\Bigr)\Bigr] + \cdots = \frac{2GM}{c^2 r^2} \qquad \forall r > R \; . \tag{65}$$

Now recall from section 2 that MOND requires $\epsilon_2 = 2$, and we have just seen that any model of the class (7) must have $\epsilon_1 = -\epsilon_2$. The weak field limit (55) for the small potential therefore implies we must have,

$$\varphi'(r) \longrightarrow -\frac{6}{r}\sqrt{\frac{a_0 GM}{c^4}} + \ldots \tag{66}$$

It follows that the constant term within the square brackets of (65) must exactly cancel, and that the next order term must involve one power of φ'. Working out the algebra gives,

$$\mathcal{F}'(x) = -\frac{1}{6} - \frac{\sqrt{x}}{108} + O(x) \Longrightarrow \mathcal{F}(x) = -\frac{x}{6} - \frac{x^{\frac{3}{2}}}{162} + O(x^2) \; . \tag{67}$$

The associated weak fields are,

$$a(r) \longrightarrow \frac{4GM}{3c^2 r} - 2\sqrt{\frac{a_0 GM}{c^4}} \;,\; b(r) \longrightarrow -\frac{8GM}{3c^2 r} + 2\sqrt{\frac{a_0 GM}{c^4}} \ln\Bigl(\frac{r}{R}\Bigr) \; . \quad (68)$$

For the general weak field ansatz (20) of section 2 we have just shown $-2\delta_1 = \delta_2 = -\frac{8}{3}$ and $-\epsilon_1 = \epsilon_2 = 2$.

A potentially troublesome point is that equation (65) involves terms of second order in the weak fields $a(r)$ and $b(r)$,

$$\varphi' \Bigl[1 + 6\mathcal{F}'\Bigl(\frac{c^4 \varphi'^2}{a_0^2}\Bigr)\Bigr] + \cdots = \frac{c^2}{18 a_0} \varphi'^2 + \cdots = \frac{2GM}{c^2 r^2} \; . \qquad (69)$$

But in (56) we previously neglected such second order terms to derive the simplified equations (57-58) which pertain in the MOND regime. Closer inspection reveals that all of the terms neglected in (56) contribute terms to the left hand side of (69) which are small for $r \ll R_{\rm hor} \sim 10^{26}$ m,

$$r \varphi'^2 \ll \frac{c^2}{a_0} \varphi'^2 \sim 10^{27} \text{ m} \times \varphi'^2 \; . \qquad (70)$$

Note also that one *must* involve quadratic terms like those of (69) in order to make the weak fields go like the square root of the system's mass, as MOND predicts.

Enforcing the MOND limit determines only the first two terms in the small x expansion of the interpolating function $\mathcal{F}(x)$. There are many ways of extending this to a formula for general x. The only phenomenological constraint is that we need the MOND corrections to be acceptably small in the general relativistic regime of large x. For example, we can make $\mathcal{F}(x) \longrightarrow -\frac{14}{3} |x|^{\frac12}$ for large $|x|$ with the following extension,

$$\begin{aligned} \mathcal{F}'(x) &= -\frac{\frac{7}{18} \mathrm{sgn}(x)}{1 + \frac16 |x|^{\frac12}} + \frac{\frac29 \mathrm{sgn}(x)}{\Bigl(1 + \frac16 |x|^{\frac12}\Bigr)^2} \;, & (71) \\ \mathcal{F}(x) &= -\frac{\frac{22}{3} |x|^{\frac12} + \frac79 |x|}{1 + \frac16 |x|^{\frac12}} + 44 \ln\Bigl(1 + \frac{1}{6} |x|^{\frac12}\Bigr) \; . & (72) \end{aligned}$$

For $|x| \gg 1$ this would typically suppress MOND corrections by some characteristic length of the system divided by $c^2/a_0 \sim 10^{27}$ m. If that is not sufficient one can always extend $\mathcal{F}(x)$ differently to obtain more suppression.

5 Homogeneous and isotropic sources

Although our relativistic formulation of MOND is ruled out by lensing it seems a pity not to work out the cosmology now that we have the formalism. The exercise also affords a potentially important caveat on just how much a general formulation of MOND can change in passing from the static geometries of galaxies to the time dependent geometry of cosmology. We begin by working out the small and large potentials for a homogeneous, isotropic and spatially flat metric,

$$ds^2 \equiv -c^2 dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} \; . \tag{73}$$

The nonlocal field equations (37) are next specialized to this geometry. Then the MOND limit is taken. The section closes by considering what happens when matter domination follows a long period of radiation domination.

In this geometry the Ricci scalar is,

$$c^2 R = 6\dot{H} + 12H^2 \qquad \text{where} \qquad H \equiv \frac{\dot{a}}{a} \; . \tag{74}$$

The small potential is defined by the equation,

$$\square \varphi(t) = -a^{-3} \frac{d}{dct}\Big(a^3 \frac{d\varphi}{dct}\Big) = R(t) \; . \tag{75}$$

If we define the initial values of φ and its first derivative to be zero, it takes the simple form,

$$\varphi(t) = -\int_0^t dt' a^{-3}(t') \int_0^{t'} dt'' a^3(t'') \Big(6\dot{H}(t'') + 12H^2(t'')\Big) \; . \tag{76}$$

The large potential is defined by the differential equation,

$$\partial_\mu \Big(\sqrt{-g} g^{\mu\nu} \Phi_{,\nu}\Big) = \partial_\mu \Big(\sqrt{-g} g^{\mu\nu} \varphi_{,\nu} \mathcal{F}'\Big) \Longrightarrow \frac{d}{dt}\Big(a^3 \dot{\Phi}\Big) = \frac{d}{dt}\Big(a^3 \dot{\varphi} \mathcal{F}'\Big) \; . \tag{77}$$

If we again assume null initial values the result is,

$$\Phi(t) = \int_0^t dt' \dot{\varphi}(t') \mathcal{F}'\Big(-c^2 a_0^{-2} \dot{\varphi}^2(t')\Big) \; . \tag{78}$$

The nonzero components of the second covariant derivative are,

$$\Phi_{;00} = c^{-2} \ddot{\Phi} \qquad , \qquad \Phi_{;ij} = -c^{-2} H \dot{\Phi} g_{ij} \; . \tag{79}$$

We assume the stress-energy tensor to take the perfect fluid form,

In the MOND regime the interpolating function and its derivative take the forms,

$$\mathcal{F}(x) \longrightarrow -\frac{1}{6}|x| \qquad , \qquad \mathcal{F}'(x) \longrightarrow -\frac{1}{6}\mathrm{sgn}(x) \; . \tag{80}$$

For cosmology the argument $x = -(c\dot{\varphi}/a_0)^2$ is *negative* so the large potential has the same sign as the small potential,

$$\Phi(t) \longrightarrow \frac{1}{6}\varphi(t) \; . \tag{81}$$

In the MOND regime we can therefore express (**??**) as,

$$-H\dot{\varphi} + 3H^2\Bigl(1 - \frac{1}{3}\varphi\Bigr) - \frac{1}{12}\dot{\varphi}^2 + \cdots = 8\pi G c^{-2}\rho \; . \tag{82}$$

An additional specialization of great interest to cosmology is the case of a power law scale factor,

$$a(t) = \Bigl(1 + H_i t\Bigr)^s \; . \tag{83}$$

Here H_i is $1/s$ times the Hubble parameter at $t = 0$. Substituting into (76) gives the small potential,

$$\varphi(t) = -6s\Bigl(\frac{2s-1}{3s-1}\Bigr)\Bigl\{\ln\Bigl[1 + H_i t\Bigr] - (1-3s)^{-1}\Bigl[\Bigl(1 + H_i t\Bigr)^{1-3s} - 1\Bigr]\Bigr\}. \tag{84}$$

At late times only the logarithm matters. In this regime we can also express $\dot{\varphi}$ in terms of the Hubble parameter,

$$\dot{\varphi}(t) \longrightarrow -6\Bigl(\frac{2s-1}{3s-1}\Bigr)H(t) \; . \tag{85}$$

We can therefore write the MOND analog of the Friedman equation for power law expansion,

$$3\Bigl\{1 + 2\sigma - \sigma^2 + 2s\sigma \ln\Bigl[1 + H_i t\Bigr]\Bigr\}H^2(t) + \cdots = 8\pi G c^{-2}\rho(t) \; , \tag{86}$$

where $\sigma \equiv (2s-1)/(3s-1)$. The effect of the logarithm on $s > \frac{1}{2}$ power laws is to gradually slow the expansion. This makes rough physical sense if we think in terms of MOND strengthening the force of gravity in the weak field regime.

For the case of radiation domination ($s = 1/2$ and $\sigma = 0$) we note that $\varphi(t) = 0$! The large potential also vanishes — exactly, not just in the MOND regime. Hence the equations reduce to those of general relativity, but with the energy and pressure coming from only ordinary matter. This is phenomenologically unacceptable. One of many things that goes wrong is nucleosynthesis.

6 Discussion

We have succeeded in embedding the MOND force law in a set of covariant, causal and conserved field equations for the metric. The model suffers from at least two fatal phenomenological problems in which its predictions agree with those of general relativity without dark matter. The deflection of light and cosmology during radiation domination both have this property. Of course these problems do not necessarily mean that MOND is wrong, only that our realization of it is.

Our model consists of corrections which are based on the small potential $\varphi[g] = \Box^{-1} R$. One naturally wonders if it is possible to find a nonlocal scalar potential that avoids the problems with lensing and cosmology while still keeping the MOND force law. For example, one might replace the covariant d'Alembertian with the conformal one,

$$\varphi_c[g] \equiv \frac{1}{\Box_c} R \qquad \text{where} \qquad \Box_c \equiv \Box - \frac{1}{6} R \; . \tag{87}$$

The distinction between $\Box$ and $\Box_c$ disappears in the weak field regime because R goes like one power of the weak fields over r^2. So such a model would still give the MOND force law. Unfortunately it would also give too little lensing precisely because its weak field limit agrees with that of $\varphi[g]$. Because φ_c vanishes with R, it would also have problems dealing with radiation domination without dark matter.

The fact that R vanishes for a radiation dominated universe means that we should avoid it as the source upon which the nonlocal operator acts. The next most complicated scalar potential would seem to be,

$$\varphi_2[g] \equiv \frac{c^4}{a_0^2} \frac{1}{\Box} \Big(R^{\mu\nu} R_{\mu\nu} \Big) \; . \tag{88}$$

Because φ_2 has roughly two derivatives acting upon two powers of the weak fields, one must also change the Lagrangian,

$$\mathcal{L}_2 = \frac{c^4}{16\pi G} \Big[R + c^{-4} a_0^2 \mathcal{F}_2 \Big(\varphi_2[g] \Big) \Big] \sqrt{-g} \; . \tag{89}$$

The interpolating function $\mathcal{F}_2(x)$ would become linear in the MOND regime.

One interesting thing about our model is that it becomes conformally invariant in the MOND limit. The first hint of this came when all our MOND corrections cancelled out of the formula for the deflection of light. Photons are conformally invariant so they experience no deflection due to conformal transformations of the metric.

To prove asymptotic conformal invariance, note that in the MOND limit ($\mathcal{F}(x) \longrightarrow -\frac{1}{6}x$ for $x > 0$) the large potential is,

$$\Phi[g] = \frac{1}{\Box} \Big(\varphi^{,\rho} \mathcal{F}' \Big)_{;\rho} \longrightarrow -\frac{1}{6} \varphi \; . \tag{90}$$

In this limit the field equations (37) take the form,

$$8\pi G c^{-4} T_{\mu\nu} = \frac{1}{3} \Big(g_{\mu\nu} \Box \varphi - \varphi_{;\mu\nu} \Big) + R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \ldots \; . \tag{91}$$

The right hand side is traceless, which means that the linearized theory is conformally invariant.

This sort of asymptotic conformal invariance was also found by Bekenstein and Milgrom [18, 27]. Conformal invariance was certainly not built into either model. For example, the trace of our full field equations (37) is nonzero,

$$8\pi G c^{-4} T^{\mu}_{\ \mu} = -6 \Box \Phi - R[1 - 2\Phi] + 2\varphi^{,\mu} \Phi_{,\mu} + \varphi^{,\mu} \varphi_{,\mu} \mathcal{F}' - \frac{2a_0^2}{c^4} \mathcal{F} \; . \tag{92}$$

Asymptotic conformal invariance arises from enforcing the MOND limit. Since MOND requires the weak fields to go as the *square root* of the source mass, it is necessary that terms linear in the weak fields drop out of some component of the field equations. The only distinguished component in a conserved, tensor equation is the trace. Hence terms linear in the weak fields must drop out of the trace of the field equations, which means that the linearized theory is conformally invariant.

If asymptotic conformal invariance is generic it means that no metric-based formulation of MOND can give the required amount of gravitational lensing. It might also bear on the view that the successes of MOND derive from galaxy formation and evolution flowing, through conventional physics, towards some sort of fixed point. This is because critical phenomena and universality are typically characterized by conformal invariance. It might be interesting if the analogy could be pursued sufficiently to generate quantitative predictions.

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